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ON THE WEAK CONTROL OF SLOWLY DAMPED SYSTEMS

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Asymptotic formulas are obtained which make it possible to derive the first approximation solution of the Riccati matrix algebraic equation of special form. Method is based on Bass' formulas [1] and the theory of perturbations [2]. The problem of control of a slowly damped oscillator is investigated in detail. Formulation of the problem in this paper differs substantially from that in [3] (no assumption is made about single-frequency oscillations, and only a stationary system is considered over an infinite time interval).

1. Statement of the problem. Motion of the controlled object is defined by the system of linear ordinary differential equations

$$\mathbf{x}^{\cdot} = F\mathbf{x} + G\mathbf{u}, \quad \mathbf{x}(0) \neq 0 \tag{1.1}$$

We have to determine vector $\mathbf{u}$ of control actions as a function of the phase vector $\mathbf{x}$ which minimizes the quadratic performance criterion

$$I = \int_0^\infty (\mathbf{x}'Q\mathbf{x} + \mathbf{u}'B_\varepsilon \mathbf{u})\, dt \tag{1.2}$$

where the prime indicates transposition, and F, G, $Q = Q'$ and $B_\varepsilon = B_\varepsilon'$ are constant matrices, with the pair F and G amenable to stabilization [1].

Matrix B_ε is assumed large. This definition is formalized by the introduction of the small parameter ε

$$B_\varepsilon = \varepsilon^{-2} B$$

The use in this problem of the term "weak control" is related to that after the formal substitution $\mathbf{u} = \varepsilon \mathbf{v}$ system (1.1) is weakly controllable in the meaning given in [4]. The term "slowly damped system" means that matrix F is nearly skew-symmetric, i.e. that $F = F_0 + \varepsilon\Phi$, $F_0 = (F - F')/2 \gg (F + F')/2 = \varepsilon\Phi$. The use of this term is explained by that the equations which define the motion of an undamped mechanical system with n degrees of freedom can be reduced to a system of $2n$ differential equations of the first order with a skew-symmetric matrix (see, e.g., [5]).

The solution of the problem of optimal control synthesis for system (1.1) that satisfies criterion (1.2) reduces to finding the solution of the Riccati matrix algebraic equation (see, e.g., [1])

$$PF + F'P - \varepsilon PGB^{-1}G'P + \varepsilon Q = 0, \quad P = \varepsilon S \tag{1.3}$$

Certain problems of determination of solid body orientation also involve investigation of Riccati equations of a similar type [6]. In what follows we assume that there exists for any $\varepsilon > 0$ a solution of (1.3) for which roots of the characteristic polynomial of matrix $F - \varepsilon GB^{-1}G'P$ lie in the left-hand half-plane.

2. Derivation of asymptotic relations. Bass' formulas [1] and the perturbation theory [2] are used below for deriving the first approximation with respect to ε of the solution of Eq. (1.3). The system of Euler's differential equations whose matrix is of the form

$$Z = \begin{Vmatrix} F_0 + \varepsilon\Phi & -\varepsilon GB^{-1}G' \\ -\varepsilon Q & F_0 - \varepsilon\Phi' \end{Vmatrix} \quad (-F_0' = F_0) \tag{2.1}$$

corresponds to Eq. (1.3).

Let $\varphi(s)$ represent the result of factorization of the characteristic polynomial of matrix (2.1)

$$\det \| Z - Es \| = \varphi(s)\ \varphi(-s) \tag{2.2}$$

with the roots of $\varphi(s)$ lying in the left-hand half-plane. Then according to [1] the sought matrix P satisfies Bass' relation

$$\varphi(Z)\begin{Vmatrix} E \\ P \end{Vmatrix} = 0 \tag{2.3}$$

Let us represent matrix Z and the polynomial $\varphi(s)$ in the form

$$Z = Z_0 + \varepsilon W, \quad Z_0 = \begin{Vmatrix} F_0 & 0 \\ 0 & F_0 \end{Vmatrix}, \quad W = \begin{Vmatrix} \Phi & -GB^{-1}G' \\ -Q & -\Phi' \end{Vmatrix} \tag{2.4}$$

$$\varphi(s) = s^n + \delta p_1 s^{n-1} + (p_{20} + \delta p_2)s^{n-2} + \delta p_3 s^{n-3} + \ldots$$
$$\ldots + (p_{n0} + \delta p_n)$$

and take into acount that for $\varepsilon = 0$

$$\varphi_{\varepsilon=0}(s) = \varphi_{\varepsilon=0}(-s) = \det \| F_0 - Es \| = s^n + p_{20}s^{n-2} + \ldots + p_{n0}$$

where the absence of odd powers of s is due to the skew-symmetry of matrix F_0 and the polynomial $\varphi_{\varepsilon=0}(s)$ has $n/2$ pairs of imaginary roots $\pm i\nu_j (j = 1, 2, \ldots, n/2))$. The quantities δp_k $(k = 1, \ldots, n)$ in (2.4) are small when ε is fairly small. Taking into account that

$$\varphi_{\varepsilon=0}(Z_0) = Z_0^n + p_{20}Z_0^{n-2} + \ldots + p_{n0}E = 0$$

we represent formula (2.3) with an accuracy to smalls of second order with respect to ε in the form

$$\Big[\Big(\sum_{k=1}^{n} Z_0^{k-1}\varepsilon W Z_0^{n-k}\Big) + \delta p_1 Z_0^{n-1} + p_{20}\Big(\sum_{k=3}^{n} Z_0^{k-3}\varepsilon W Z_0^{n-k}\Big) + \tag{2.5}$$

$$\delta p_3 Z_0^{n-3} + \cdots + \delta p_n E\Big]\left\|\begin{matrix} E \\ P \end{matrix}\right\| = 0$$

Formula (2.5) shows that the basic difficulty in using Bass' formula (2.3) relates to the necessity of a reasonably accurate determination of coefficients of the polynomial $\varphi(s)$ (or of corrections δp_k to the coefficients). The problem can be considered solved when corrections to roots $\pm i\nu_j$ of polynomial $\varphi_{\varepsilon=0}(s)$ have been determined with reasonable accuracy. Let us determine these corrections. We seek the roots μ_l of the characteristic polynomial of matrix Z in the form of series in powers of ε

$$\begin{aligned} &l = 2k - 1, \quad \mu_l = i\nu_k + \varepsilon\lambda_{1l} + O(\varepsilon^2) \\ &l = 2k, \quad \mu_l = -i\nu_k + \varepsilon\lambda_{1l} + O(\varepsilon^2) \quad (k = 1, 2, \ldots) \end{aligned} \qquad (2.6)$$

Corrections $\varepsilon\lambda_{1l}$ cannot be determined hy the direct application of results of the perturbation theory [2] to matrix Z, since Z_0 is not a self-conjugate transformation. Because of this we consider matrix

$$Z^2 = Z_0^2 + Z_0\varepsilon W + \varepsilon W Z_0 + (\varepsilon W)^2 = Z_0^2 + \varepsilon T$$

to which it is possible to apply the results of [2], since matrix Z_0^2 is symmetric. The roots of characteristic polynomials of matrices Z_0^2 and Z^2 are, respectively, $-\nu_j^2$ and μ_l^2

$$\begin{aligned} &l = 2k - 1, \quad \mu_l^2 = -\nu_k^2 + \varepsilon\gamma_{1l} + O(\varepsilon^2), \quad \varepsilon\gamma_{1l} = 2i\lambda_{1l}\nu_k \\ &l = 2k, \quad \mu_l^2 = -\nu_k^2 + \varepsilon\gamma_{1l} + O(\varepsilon^2), \quad \varepsilon\gamma_{1l} = -2i\lambda_{1l}\nu_k \end{aligned} \qquad (2.7)$$

where the corrections $\varepsilon\gamma_{1l}$ can be determined by the method of the perturbation theory.

Let us determine $\varepsilon\gamma_{1l}$. We assume that $-\nu_k^2$ is a $2r$-multiple eigenvalue of matrix Z_0^2. We denote by $\mathbf{f}_1^k, \mathbf{f}_2^k, \ldots, \mathbf{f}_{2r}^k$ the set of orthonormal eigenvectors of matrix Z_0^2 that correspond to $-\nu_k^2$. According to [2] corrections $\varepsilon\gamma_{1k}$ are eigenvalues of matrix $D_k = \| d_{mn}^k \|$ whose elements are scalar products of vectors $\varepsilon T\mathbf{f}_n^k$ and $\mathbf{f}_m^k$

$$d_{mn}^k = (\varepsilon T\mathbf{f}_n^k \cdot \mathbf{f}_m^k)$$

Hence corrections $\varepsilon\gamma_{1k}$ are roots of the equation

$$\det \| D_k - \varepsilon\gamma_{1k} E \| = 0 \qquad (2.8)$$

Formulas (2.6) and (2.8) make possible the determination of roots of matrix Z in the first approximation and, consequently, also corrections δp_k to coefficients of polynomial $\varphi(s)$. If the first approximation corrections are such that all roots μ_l have nonzero real parts, formulas (2.5) makes it possible to find the approximate solution of Eq. (1.3). Since for the obtained approximate value of P the roots of the characteristic polynomial of matrix $F - \varepsilon G B^{-1} G' P$ lie in the left-hand half-plane (they coincide with those roots μ_l $(l = 1, \ldots, n)$ whose real parts are negative), hence using the Newton —Rafson scheme [1] it is possible to further refine the obtained value of P. If, however, in the first approximation there are imaginary roots among μ_l, subsequent approximations must be used for determining P.

3. **Approximate solution of the problem of control of a single oscillator.** The equations of motion of the controlled object are of the form

$$x^{\cdot} = y, \quad y^{\cdot} = -x - \varepsilon\beta y + u \tag{3.1}$$

We define the performance criterion by formula

$$I = \int_0^\infty (q_1 x^2 + q_2 y^2 + \varepsilon^{-2} u^2)\, dt \tag{3.2}$$

Matrices in Eq. (1.3) and subsequent relationships are of the form

$$F = \left\| \begin{matrix} 0 & 1 \\ -1 & -\varepsilon\beta \end{matrix} \right\|, \quad G = \left\| \begin{matrix} 0 \\ 1 \end{matrix} \right\|, \quad B_\varepsilon = \varepsilon^{-2}, \quad B = 1$$

$$Q = \left\| \begin{matrix} q_1 & 0 \\ 0 & q_2 \end{matrix} \right\|, \quad GB^{-1}G' = \left\| \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} \right\|, \quad F_0 = \left\| \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \right\|$$

$$\Phi = \left\| \begin{matrix} 0 & 0 \\ 0 & -\beta \end{matrix} \right\|, \quad Z_0 = \left\| \begin{matrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{matrix} \right\|, \quad W = \left\| \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & -\beta & 0 & -1 \\ -q_1 & 0 & 0 & 0 \\ 0 & -q_2 & 0 & \beta \end{matrix} \right\|$$

For $\varepsilon = 0$ the polynomial $\varphi(s)$ is defined by

$$\varphi_{\varepsilon=0}(s) = \det \| F_0 - Es \| = s^2 + 1$$

Since the roots of this polynomial are $\pm i$, hence $p_{20} = 1$ and $\nu = 1$. Let us determine the corrections to the zero approximation of roots of matrix Z.

Matrix $Z_0^2 = -E$ has 1 as its unique eigenvalue whose multiplicity is four. Vectors

$$\mathrm{f}_1 = \left\| \begin{matrix} 1 \\ 0 \\ 0 \\ 0 \end{matrix} \right\|, \quad \mathrm{f}_2 = \left\| \begin{matrix} 0 \\ 1 \\ 0 \\ 0 \end{matrix} \right\|, \quad \mathrm{f}_3 = \left\| \begin{matrix} 0 \\ 0 \\ 1 \\ 0 \end{matrix} \right\|, \quad \mathrm{f}_4 = \left\| \begin{matrix} 0 \\ 0 \\ 0 \\ 1 \end{matrix} \right\|$$

can be chosen as the set of orthonormal vectors corresponding to that eigenvalue.

Matrix $D = \| d_{nm} \|$ is equal εT; it can be defined with an accuracy to ε^2 in the form

$$D = \varepsilon T \approx \varepsilon W Z_0 + Z_0 \varepsilon W = \varepsilon \left\| \begin{matrix} 0 & -\beta & 0 & -1 \\ \beta & 0 & 1 & 0 \\ 0 & -(q_1 + q_2) & 0 & \beta \\ (q_1 + q_2) & 0 & -\beta & 0 \end{matrix} \right\|$$

Note that matrix D can be represented in the form of the Kronecker product of two second order matrices

$$\varepsilon W Z_0 + Z_0 \varepsilon W = \begin{Vmatrix} 0 & -\varepsilon \\ \varepsilon & 0 \end{Vmatrix} \times \begin{Vmatrix} \beta & 1 \\ (q_1 + q_2) & -\beta \end{Vmatrix}$$

hence the eigenvalues of matrix D (corrections $\varepsilon\gamma_{1l}$) are the products of eigenvalues of matrix cofactors whose eigenvalues are $\pm i\varepsilon$ and $\pm \sqrt{\beta^2 + q_1 + q_2}$. Thus

$$\varepsilon\gamma_{1l} = \pm i\varepsilon \sqrt{\beta^2 + q_1 + q_2}, \qquad \varepsilon\lambda_{1l} = \pm {}^1/_2\, \varepsilon \sqrt{\beta^2 + q_1 + q_2}$$

and, consequently, the first approximation roots μ_1 and μ_2 of polynomial $\varphi(s)$ are

$$\mu_{1,2} = -{}^1/_2\, \varepsilon \sqrt{\beta^2 + q_1 + q_2} \pm i \tag{3.3}$$

The same asymptotic formula for roots $\mu_{1,2}$ can be obtained using the results [of analysis] in [7], according to which in the notation used here the following relations between coefficients q_1 and q_2, roots μ_1 and μ_2 of polynomial $\varphi(s)$, and roots θ_1 and θ_2 of the characteristic polynomial of matrix F:

$$\begin{aligned} \varepsilon^2 q_1 &= (\mu_1\mu_2)^2 - (\theta_1\theta_2)^2 \\ \varepsilon^2 q_2 &= (\mu_1 + \mu_2)^2 - (\theta_1 + \theta_2)^2 + 2(\theta_1\theta_2 - \mu_1\mu_2) \end{aligned} \tag{3.4}$$

are valid. Let

$$\mu_{1,2} = -(\varepsilon\rho_1 + \varepsilon^2\rho_2 + O(\varepsilon^3)) \pm i(1 + \varepsilon\eta_1 + \varepsilon^2\eta_2 + O(\varepsilon^3))$$

Since

$$\theta_{1,2} = -\varepsilon\beta/2 \pm \sqrt{\varepsilon^2\beta^2/4 - 1}$$

hence with an accuracy within ε^2 from (3.4) we have

$$\eta_1 = 0, \quad q_1 = 2\rho_1^2 + 4\eta_2, \quad q_2 = 2\rho_1^2 - 4\eta_2 - \beta^2$$

Hence the approximate expression for roots $\mu_{1,2}$ is of the form (3.3) accurate within ε^2.

Let us now determine the corrections of coefficients δp_1 and δp_2

$$\begin{aligned} \delta p_1 &= -(\mu_1 + \mu_2) = \varepsilon \sqrt{\beta^2 + q_1 + q_2} \\ \delta p_2 &= \mu_1\mu_2 - p_{20} = \mu_1\mu_2 - 1 = \varepsilon^2(\beta^2 + q + q_2)/4 \end{aligned}$$

Formula (2.5) accurate to within smalls of second order is of the form

$$[\varepsilon W Z_0 + Z_0 \varepsilon W + \delta p_1 Z_0] \begin{Vmatrix} E \\ P \end{Vmatrix} = 0$$

This expression is concretely defined by the following two matrix equations :

$$\begin{Vmatrix} 0 & -\Lambda_- \\ \Lambda_- & 0 \end{Vmatrix} + \begin{Vmatrix} 0 & -\varepsilon \\ \varepsilon & 0 \end{Vmatrix} P = 0$$

$$\left\| \begin{matrix} 0 & -\varepsilon(q_1+q_2) \\ \varepsilon(q_1+q_2) & 0 \end{matrix} \right\| + \left\| \begin{matrix} 0 & \Lambda_+ \\ -\Lambda_+ & 0 \end{matrix} \right\| P = 0$$

$$\Lambda_\pm = \varepsilon\beta \pm \varepsilon \sqrt{\beta^2 + q_1 + q_2}$$

each of which can be used for determining matrix P.

Thus the first approximation of the sought solution of Eq. (1.3) for the considered here problem is of the form

$$P = -\Lambda_- E\varepsilon^{-1}, \quad S = -\Lambda_- E\varepsilon^{-2} \tag{3.5}$$

Let us evaluate the quantity of the approximation obtained by these formulas. Let the damping in system (3.1) be fixed, i.e. $0 < \varepsilon\beta = \beta_0 = \text{const}$. The Liapunov equation into which the Riccati equation (1.3) is transformed for $\varepsilon = 0$ has the solution

$$S = \frac{q_1+q_2}{2\beta_0} E + \frac{q_1}{2} \left\| \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \right\| + \frac{\beta_0 q_1}{2} \left\| \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} \right\|$$

The approximate expression for S in (3.5) approaches the first term (which for small β_0 is the principal one) of this solution when $\varepsilon \to 0$, since

$$\lim_{\varepsilon \to 0} S = \lim_{\varepsilon \to 0} \frac{-\beta_0 + \sqrt{\beta_0^2 + \varepsilon^2 q_1 + \varepsilon^2 q_2}}{\varepsilon^2} E = \frac{q_1+q_2}{2\beta_0} E$$

In the absence of damping in system (3.1), i.e. $\beta = 0$, Eq. (1.3) is satisfied with an accuracy within ε by any matrix which is a multiple of the unit matrix ($P = aE$). This becomes clear if we substitute in (2.1) matrix aE for P, which yields the following formula for the discrepancy matrix:

$$aF + aF' - a^2\varepsilon GB^{-1}G' + \varepsilon Q = \varepsilon \left\| \begin{matrix} q_1 & 0 \\ 0 & q_2 - a^2 \end{matrix} \right\|$$

Note that the coefficient $a = \sqrt{q_1 + q_2}$, which corresponds to (3.5) does not, generally speaking, minimize the norm of the discrepancy matrix, although even in this case formula (3.5) may yield a good approximation. Thus, for example, for $q_1 = 1$, $q_2 = 3$, $\varepsilon = 0.1$, and $\beta = 0$, from (3.5) we have

$$P = \left\| \begin{matrix} 2 & 0 \\ 0 & 2 \end{matrix} \right\|$$

The exact solution of Eq. (2.1) is in this case of the form

$$P = \left\| \begin{matrix} 2.01 & 4.99\cdot 10^{-2} \\ 4.99\cdot 10^{-2} & 2.0 \end{matrix} \right\|$$

Note that for $\beta = 0$ the asymptotic formula for P that coincides with (3.5) (with accuracy within ε) may be derived from the exact solution of this problem (see [8], Example 2)

$$P = A\{C + \alpha H\}^{-1}, \qquad A = \left\| \begin{matrix} \varepsilon q_1 & 0 \\ 0 & \varepsilon q_2 \end{matrix} \right\|, \qquad C = \left\| \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \right\|$$

$$H = \left\| \begin{matrix} -q_1 a & -bd^{-1} \\ b & -q_2 a \end{matrix} \right\|, \quad a = \frac{\varepsilon^2 (q_1 + q_2 d)}{4d\tau}$$
$$b = {}^1/_4 \varepsilon^2 \tau^{-2} (q_1 + q_2)(q_1 + q_2 d), \qquad \alpha = d\,[\varepsilon q_1 q_2 / 4 + {}^1/_4 \varepsilon^2 \tau^{-2} (q_1 + q_2)^2]^{-1},$$
$$\tau = \sqrt{\varepsilon^2 q_2 + 2d - 2}, \qquad d = \sqrt{1 + \varepsilon^2 q_1}$$

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